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Approximate Expansion for Function Theoretic Representation of Solutions of the Helmholtz Equation

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PREFACE

This work reported in this document was completed under NUSC Project No. A92045, Dr. Rolf Kasper, Principal Investigator.

A portion of this work was completed while M. D. Duston participated in the U.S. Navy-ASEE Summer Faculty Research Program at the NUSC, New London Laboratory, in conjunction with NUSC Independent Research Project A92045 and while R. P. Gilbert was on an Intergovernmental Personnel Act Mobility Assisgnment to NUSC from the University of Delaware.

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This document is based on a presentation given at the American Mathematical Society National Meeting, New Orleans, Louisiana, January 1986.

We start from a function theoretic (transmutation) representation of the solutions of the class of Helmholtz equations that have coefficients that vary in one direction and satisfy a radiation condition in orthogonal directions. The kernal of the required transmutation operator satisfies a mixed Cauchy-Gorsat problem for a hyperbolic partial differential equation in two variables. We present an expansion of the kernal function that can be truncated to produce approximations that are suitable for applications of the desired transmutations, and we compare to other approximation techniques.

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APPROXIMATE EXPANSION FOR FUNCTION THEORETIC REPRESENTATION OF SOLUTIONS OF THE HELMHOLTZ EQUATION

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BY APPLYING SEPARATION OF VARIABLES TO THE HELMHOLTZ EQUATION, WE FIND THAT WE WANT TO TRANSMUTE SOLUTIONS OF

$$\frac{d^2}{dz^2}\varphi + k^2\varphi = \lambda\varphi$$

INTO SOLUTIONS OF

$$\frac{d^2}{dz^2}\psi + k^2[1 + \varepsilon s(z)]\psi = \lambda\psi.$$

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VUGRAPH 1

We are primarily interested in a problem from underwater acoustics. By a "function theoretic method" we mean a transmutation of the Helmholtz equation with constant coefficients into solutions of the Helmholtz equation with variable coefficients. The Helmholtz equation is a partial differential equation governing the propagation of sound. We start with a model of the ocean using the depth dependent Helmholtz equation, and apply separation of variables. The depth-dependent part of the separated Helmholtz equation with constant coefficients satisfies the ordinary differential equation

$$\frac{d^2}{dz^2}\phi + k^2\phi = \lambda\phi .$$

This represents an ocean with uniform sound speed. The depth dependent part of the Helmholtz equation with variable coefficients satisfies the ordinary differential equation:

$$\frac{d^2}{dz^2}\psi + k^2[1 + \epsilon s(z)]\psi = \lambda\psi ,$$

which represents an ocean where the sound speed is a function of the depth. Here the term $\epsilon s(z)$ is the perturbation of the index of refraction of the variable coefficient Helmholtz equation. The small parameter ϵ is a measure of the size of the perturbation.

THE TRANSMUTATION OPERATION COMPUTES

$$\psi(z) = \varphi(z) + \int_{z_b}^{z} K(z,s)\, \varphi(s)\, ds$$

IN TERMS OF THE TRANSMUTATION KERNEL K(z,s) WHICH MUST SATISFY

$$\frac{\partial^2 K}{\partial z^2} - \frac{\partial^2 K}{\partial s^2} + k^2 \epsilon s(z)\, K = 0.$$

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VUGRAPH 2

We are looking for a transmutation of the form

$$\psi(z) = \phi(z) + \int_{z_b}^{z} K(z,s)\phi(s)\, ds \;.$$

where the kernel function K(z,s) must be determined. We substitute the transmutation into the ordinary differential equation for the separated variable coefficient Helmholtz equation, and this yields a partial differential equation for the kernel function K(z,s)

$$\frac{\partial^2 K}{\partial z^2} - \frac{\partial^2 K}{\partial s^2} + k^2 \epsilon s(z) K = 0 \;.$$

Although this may seem to be stepping backwards after going through the trouble of reducing the Helmholtz equation to an ordinary differential equation problem we will demonstrate a method of approximating the kernel by iterative integrals.

BUT WE NEED BOUNDARY CONDITIONS.

FOR EXAMPLE, SUPPOSE

$\psi(0) = 0$ **AND** $\frac{\partial}{\partial z}\psi(z_b) = 0.$

SINCE

$$\psi(z) = \varphi(z) + \int_{z_b}^{z} K(z,s)\,\varphi(s)\,ds,$$

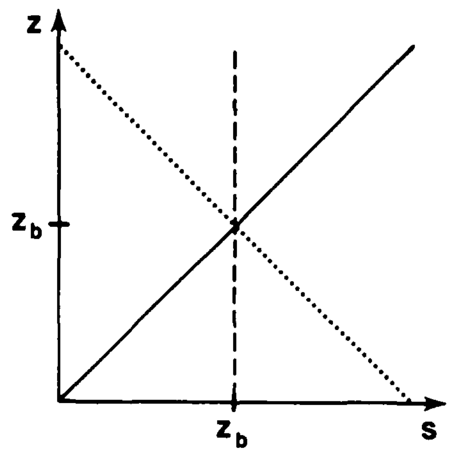
WE ASK

$$\frac{\partial}{\partial z}\varphi(z_b) = 0,$$

$$\frac{\partial}{\partial s}K(z,z_b) = 0,$$

AND

$$2\frac{\partial}{\partial z}K(z,z) + k^2\,\varepsilon s(z) = 0.$$



VUGRAPH 3

We now invoke our boundary conditions for the Helmholtz equation with variable coefficients. We ask that the pressure goes to zero at the surface of the ocean (a pressure release condition), which is $\psi(0)=0$, and that the boundary beneath the ocean and its bottom at the depth z_b be rigid, which is the condition $\psi'(z_b)=0$. We impose a similar bottom boundary condition on the solutions of the Helmholtz equation with constant coefficients. Combining this with the given transmutation we obtain two additional constraints on the kernel function $K(z,s)$,

$$\frac{\partial}{\partial s}K(z,z_b) = 0 \qquad \text{and} \qquad 2\frac{\partial}{\partial z}K(z,z) + k^2\epsilon s(z) = 0 .$$

These two conditions and the partial differential equation for $K(z,s)$ are sufficient to uniquely determine the kernel function.

ONE METHOD OF SOLVING FOR THE KERNEL, K(z,s):

LET $M(\xi,\eta) = K(z,s)$ **WHERE** $2\xi = z + s - 2z_b$, $2\eta = z - s$.

THEN

$$2\,M_{\xi\eta} + k^2\epsilon s(\xi + \eta + z_b)\,M = 0.$$

EXPAND $M(\xi,\eta)$ **AS**

$$M(\xi,\eta) = \sum_{p=1}^{\infty} k^{2p} M^{(p)}(\xi,\eta), \text{ WHERE}$$

$$2M^{(1)}(\xi,\eta) = -\int_0^{\xi} \epsilon s(\alpha + z_b)\,d\alpha - \int_0^{\eta} \epsilon s(\beta + z_b)\,d\beta$$

AND

$$M^{(p+1)}(\xi,\eta) = -\int_0^{\eta}\int_0^{\xi} M^{(p)}(\alpha,\beta)\,\epsilon s(\alpha + \beta + z_b)\,d\alpha d\beta.$$

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VUGRAPH 4

One method of solving for the kernel function $K(z,s) = M(\xi,\eta)$ is to transform into characteristic coordinates given by

$$2\xi = z + s - 2z_b \qquad \text{and} \qquad 2\eta = z - s\ .$$

We then get a partial differential equation for $M(\xi,\eta)$ which is

$$2\,M_{\xi\eta} + k^2\epsilon s(\xi + \eta + z_b)M = 0.$$

We next do a Born expansion of M in the parameter k^2. This yields a system of integrals where the first coefficient of the expansion is given by

$$2\,M^{(1)}(\xi,\eta) = -\int_0^{\xi}\epsilon s(\alpha+z_b)\,d\alpha - \int_0^{\eta}\epsilon s(\beta+z_b)\,d\beta\ ,$$

and succesive coefficients in the expansion are given by

$$M^{(p+1)}(\xi,\eta) = -\int_0^{\eta}\int_0^{\xi} M^{(p)}(\alpha,\beta)\ \epsilon s(\alpha+\beta+z_b)\ d\alpha\ d\beta\ .$$

INVERSION OF COORDINATES IN $M(\xi,\eta)$ GIVES

$$K(z,s) = \sum_{p=1}^{\infty} k^{2p} K^{(p)}(z,s)$$

$$2K^{(1)}(z,s) = -\int_0^{\frac{z+s-2z_b}{2}} \epsilon s(\alpha + z_b)\, d\alpha - \int_0^{\frac{z-s}{2}} \epsilon s(\beta + z_b)\, d\beta$$

AND

$$K^{(p+1)}(z,s) = -\int_0^{\frac{z-s}{2}} \int_0^{\frac{z+s-2z_b}{2}} K^{p}(\alpha, \beta)\, \epsilon s(\alpha + \beta + z_b)\, d\alpha\, d\beta.$$

VUGRAPH 5

Inverting coordinates from (ξ,η) back into (z,s) we obtain the expansion of the kernel function $K(z,s)$. The first term of the the expansion is

$$2\, K^{(1)}(z,s) = -\int_0^{(z+s-2z_b)/2} \epsilon s(\alpha+z_b)\, d\alpha \; -\int_0^{(z-s)/2} \epsilon s(\beta+z_b)\, d\beta\ ,$$

and the succesive terms of the expansion are given by

$$K^{(p+1)}(z,s) = -\int_0^{(z-s)/2} \int_0^{(z+s-2z_b)/2} K^{(p)}(\alpha,\beta)\ \epsilon s(\alpha+\beta+z_b)\ d\alpha\ d\beta\ .$$

We have now obtained explicit formulas for the power series expansion of the kernel $K(z,s)$ in the parameter ϵ.

EXPAND IN SERIES IN ε

$$\phi(z) = \phi^{(0)}(z) + \varepsilon\phi^{(1)}(z) + \varepsilon^2\phi^{(2)}(z) + \cdots$$

$$\psi(z) = \psi^{(0)}(z) + \varepsilon\psi^{(1)}(z) + \varepsilon^2\psi^{(2)}(z) + \cdots$$

$$\lambda = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + \cdots.$$

VUGRAPH 6

We also expand in power series in the parameter ϵ: 1) the solution of the depth dependent part of the Helmholtz equation with constant coefficients, 2) the solution of the depth dependent part of the separated Helmholtz equation with variable coefficients and 3) the constant λ.

USE THE BOUNDARY CONDITION

$$0 = \psi(0) = \varphi(0) + \int_{z_b}^{0} K(0,s)\, \varphi(s)\, ds.$$

DEFINE $\theta(z) = \varphi(z) - \varphi(0)$.

NOTICE THAT

$$\theta(0) = 0 \text{ AND } \frac{\partial}{\partial z}\theta(z_b) = 0.$$

ITS EXPANSION IS

$$\theta(z) = \theta^{(0)}(z) + \varepsilon\theta^{(1)}(z) + \varepsilon^2\theta^{(2)}(z) + \cdots.$$

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VUGRAPH 7

By examiningthe surface boundary condition we see that in general the transmutation does not allow the surface boundary condition for the separated Helmholtz equation with constant coefficients to be identical to that for the problem with variable coefficients i.e. $\phi(0)\neq 0$. Therefore we define a new function

$$\theta(z)= \phi(z) - \phi(0)$$

and see that this function does satisfy the same boundary conditions as $\psi(z)$. We then expand this new function in power series in ϵ.

ORDER ε^0:

$$\theta^{(0)''} + (\mathbf{k}^2 - \lambda^{(0)})\, \theta^{(0)} = \mathbf{0}$$

ORDER ε:

$$\theta^{(1)''} + (\mathbf{k}^2 - \lambda^{(0)})\, \theta^{(1)} - \lambda^{(1)}\theta^{(0)} = (\lambda^{(0)} - \mathbf{k}^2)\, \varphi^{(1)}(\mathbf{0})$$

ORDER ε^2:

$$\theta^{(2)''} + (\mathbf{k}^2 + \lambda^{(0)})\, \theta^{(2)} - \lambda^{(1)}\theta^{(1)} - \lambda^{(2)}\theta^{(0)} = (\lambda^{(0)} - \mathbf{k}^2)\varphi^{(2)}(\mathbf{0}) + \lambda^{(1)}\varphi^{(1)}(\mathbf{0})$$

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VUGRAPH 8

Substituting $\theta(z)$ into the ordinary differential equation for the depth dependent part of the Helmholtz equation with constant coefficients and isolating the terms with corresponding powers in ϵ we obtain a sequence of ordinary differential equations each with the same boundary conditions as the original problem.

For the 0^{th} order term it is well known that this homogeneous ordinary differential equation has a set of solutions (normalized sine functions) for $\theta^{(0)}$ and corresponding eigenvalues for $\lambda^{(0)}$.

We then take inner products of, $\theta^{(1)}$ with the 0^{th} order equation and $\theta^{(0)}$ with the first order equation and subtract to get the expression

$$\lambda^{(1)} \int_0^{z_b} \theta^{(0)}(s)\, ds = \lambda^{(1)} = \int_0^{z_b} (\lambda^{(0)} - k^2)\phi^{(1)}(0)\; \theta^{(0)}(s)\, ds\ ;$$

which yields $\lambda^{(1)}$ if we know the value of the constant $\phi^{(1)}(0)$.

THE CORRECTIONS TO FIRST ORDER IN ϵ ARE DEFINED IN TERMS OF

$$\phi_n^{(1)}(0) = \int_0^{z_b} K^{(1)}(0,s)\, \theta_n^{(0)}(s)\, ds$$

AND

$$\epsilon K^{(1)}(z,s) = -\frac{k^2}{2}\left(\int_0^{\frac{z+s-z_b}{2}} [n^2(\zeta + z_b) - 1]\, d\zeta + \int_0^{\frac{z-s}{2}} [n^2(\zeta + z_b) - 1]\, d\zeta\right).$$

WE OBTAIN

$$\lambda^{(1)} = \left[\frac{(2n-1)\pi}{2z_b} \sqrt{\frac{2}{z_b}}\ \phi_n^{(1)}(0)\right]$$

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VUGRAPH 9

We obtain the value of the constant $\phi^{(1)}(0)$ by using the power series expansion in the transmutation equation and evaluating at z=0, this gives

$$\phi^{(1)}(0) = -\int_0^{z_b} K^{(1)}(0,s)\ \theta^{(0)}(s)\ ds\ .$$

We next give an explicit formula for $\lambda^{(1)}$ in terms of the known expressions for $\lambda^{(0)}$, $\theta^{(0)}$, $\phi^{(1)}(0)$ and the expansion of the kernel function.

$$\lambda^{(1)} = \left[\frac{(2n-1)\pi}{2z_b} \sqrt{\frac{2}{h}}\ \ \phi_n^{(1)}(0)\right]$$

Integrating by parts and making a change of variables we can show that the result reduces to

$$\lambda^{(1)} = \frac{2k^2}{z_b} \int_0^{z_b} \epsilon s(\xi) \left[\sin\frac{(2m-1)\pi\xi}{2z_b}\right]^2 d\xi$$

which is the same result given by Titchmarsh in his classical application of perturbation theory.

TO FIND $\theta^{(1)}(z)$ WE RECALL

$$\theta^{(1)''} + (k^2 - \lambda^{(0)})\,\theta^{(1)} = \lambda^{(1)}\theta^{(0)} + (\lambda^{(0)} - k^2)\ \phi^{(1)}(0).$$

APPLYING VARIATIONS OF PARAMETERS,

$$\theta_p^{(1)}(z) = \int_0^{z_b} \frac{\theta^{(0)}(z-s)}{\sqrt{k^2 - \lambda^{(0)}}}\,(\lambda^{(1)}\theta^{(0)}(s) + (\lambda^{(0)} - k^2)\phi^{(1)}(0))\,ds,$$

THIS GIVES A SOLUTION

$$\theta_n^{(1)}(z) = C_1\,\theta_n^{(0)}(z) + \phi_n^{(1)}(0)\left(\left|1 - \frac{z}{z_b}\right| \cos\frac{(2n-1)\pi z}{2z_b}\right) + \phi_n^{(1)}(0),$$

WHERE THE CONSTANT C_1 MUST BE DETERMINED.

VUGRAPH 10

Going back to the ordinary differential equation for θ and the first order in ϵ expansion, we must solve an inhomogeneous ordinary differential equation

$$\theta^{(1)''} + (k^2 - \lambda^{(0)})\theta^{(1)} = \lambda^{(1)}\theta^{(0)} + (\lambda^{(0)} - k^2)\phi^{(1)}(0)$$

for $\theta^{(1)}$. We solve this by variation of parameters and obtain an explicit formula for the coefficient of the power series expansion. For general solutions to the homogeneous equation we use

$$y_1 = \sqrt{\frac{2}{z_b}}\sin\frac{(2n-1)\pi z}{2z_b} \quad \text{and} \quad y_2 = \sqrt{\frac{2}{z_b}}\cos\frac{(2n-1)\pi z}{2z_b}$$

The solution is a linear combination of these functions and the particular solution given by

$$\theta_p^{(1)} = \int_0^{z_b} \frac{\theta^{(0)}(z-s)}{\sqrt{k^2 - \lambda^{(0)}}}\,(\lambda^{(1)}\theta^{(0)}(s) + (\lambda^{(0)} - k^2)\phi^{(1)}(0))\,ds$$

The form of the solution therefore is $\theta^{(1)} = c_1y_1 + c_2y_2 + \theta_p^{(1)}$. The boundary condition at the surface $\theta^{(1)}(0)$ gives $c_2=0$. Because the transmutation preserves the boundary condition $\theta^{(1)'}(z_b)=0$, c_1 cannot be determined from this boundary condition.

USING THE RELATIONS

$$\phi_n^{(1)}(z) = \theta_n^{(1)}(z) + \phi_n^{(1)}(0)$$

AND

$$\psi_n^{(1)}(z) = \phi_n^{(1)}(z) + \int_{z_b}^{z} K^{(1)}(z,s)\,\theta^{(0)}(s)\,ds$$

WE APPLY L_2 NORMALIZATION TO OBTAIN A UNIQUE SOLUTION (C_1 IS DETERMINED)

$$\psi_n^{(1)}(z) = \Big| \phi_n^{(1)}(0) \Big(\Big|1 - \frac{z}{z_b}\Big| \cos\frac{(2n-1)\pi z}{2z_b} - \int_0^{z_b} \Big|1 - \frac{z}{z_b}\Big| \cos\frac{(2n-1)\pi z}{2z_b}\,\theta_n^{(0)}(z)\,dz \Big)$$

$$+ \Big(\int_{z_b}^{z} K^{(1)}(z,s)\,\theta_n^{(0)}(s)\,ds - \int_0^{z_b} \Big| \int_{z_b}^{z} K^{(1)}(z,s)\,\theta_n^{(0)}(s)\,ds \Big|\, \theta_n^{(0)}(z)\,dz \Big) \Big|.$$

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VUGRAPH 11

We substitute the function $\theta^{(1)}$ into the equation that determines $\phi^{(1)}$ and use this with the transmutation to determine the first order in ϵ coefficient of the power series expansion of ψ. We ask that this solution of ψ to first order in ϵ be L_2 normalized. It is this normalization constraint that allows us to give a unique, explicit formula for the first order coefficient.

THE TRANSMUTATION RESULT

$$\psi_n^{(1)}(z) = \left| \phi_n^{(1)}(0) \left(\left|1 - \frac{z}{z_b}\right| \cos\frac{(2n-1)\pi z}{2z_b} - \int_0^{z_b} \left|1-\frac{z}{z_b}\right| \cos\frac{(2n-1)\pi z}{2z_b}\,\theta_n^{(0)}(z)\,dz\right)\right.$$

$$\left. + \left(\int_{z_b}^{z} K^{(1)}(z,s)\,\theta_n^{(0)}(s)\,ds - \int_0^{z_b}\left|\int_{z_b}^{z} K^{(1)}(z,s)\,\theta_n^{(0)}(s)\,ds\right|\theta_n^{(0)}(z)\,dz\right)\right|,$$

COMPARED TO THE CLASSICAL RESULT

$$\psi_n^{(1)}(z) = \sum_{p \neq n}^{\infty} \frac{\int_0^{z_b} s(\sigma)\,\theta_n^{(0)}(\sigma)\,\theta_p^{(0)}(\sigma)\,d\sigma}{\lambda_p^{(0)} - \lambda_n^{(0)}}\,\theta_p^{(0)}(z).$$

VUGRAPH 12

We compare this transmutation result to the classical result of Titchmarsh where this coefficient is given only in terms of an infinite Fourier series.

This points to an important difference between the two approaches. In practical application both approaches may suffer inaccuracy due to truncation of the power series in ϵ; however the transmutation method does not suffer from any error due to truncation of an infinite Fourier series expansion. The transmutation term is in fact the evaluation of the infinite sum.